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Coherent states sometimes look like squeezed states, and visa versa: The Paul trap[1]

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Abstract. Using the Paul Trap as a model, we point out that the same wave functions can be variously coherent or squeezed states, depending upon the system they are applied to.

1 Introduction

Elsewhere [1]-[3], we have investigated time-dependent Schrödinger equations that are quadratic in position, q, and momentum, $p = -i\partial_q$:

$$\begin{aligned} \{2i\partial_t &- [1+g_2(t)]\,p^2 + g_0(t)\tfrac{1}{2}(qp+pq) \\ &+ g_1(t)p - 2h_2(t)q^2 - 2h_1(t)q - 2h_0(t)\Big\}\,\Psi(q,t) = 0. \end{aligned} \tag{1}$$

We use dimensionless variables ($\hbar = m = 1$) and the g's and h's are time-dependent functions. This study yielded methods for obtaining the symmetries of these systems. It also provided explicit analytic relationships between various subclasses of these equations. From that we could obtain implicit and sometimes explicit analytic solutions for the number, coherent, and squeezed states, and also for the associated uncertainty relations.

[1]Our discussion of the physical distinction between coherent and squeezed states is most appropriate as a vehicle for us to pay homage to the late Dan Walls. Since the time of their experimental discovery in the mid 1980s, his name has been closely associated with the physics of squeezed states.

In this paper, we will apply the techniques so obtained to the Paul trap system. We will thereby be able to provide insight into what are the coherent and squeezed states of the Paul trap, and into what uncertainty relations are satisfied.

In Section 2, we will set up the formalism for the particular subclass of Eq. (1) that contains the Paul trap as a special case. That is: all $h_i(t) = g_i(t) = 0$ except $h_2(t) \equiv f(t)$. We go on, in Section 3, to describe the coherent states, squeezed states, and uncertainty relations for this subclass. The physical Paul trap is described in Section 4.

In Section 5, we discuss coherent states, squeezed states, and uncertainty relations for the Paul trap. This is the main thrust of this paper. Specifically, the reader will see how the same physical states can be viewed either as coherent states of the Paul trap or as squeezed states of the harmonic oscillator, and vice-versa.

2 Time-Dependent Quadratic Systems

We now concentrate on the aforementioned particular subclass of Eq. (1), the time-dependent, harmonic-oscillator, Schrödinger equation. We have that[2]

$$\left\{\partial_{qq} + 2i\partial_t - 2f(t)q^2\right\}\Psi(q,t) = 0. \tag{2}$$

Here the function $f(t)$ is a continuously differentiable and integrable function of t. From the general results [1]-[3] for Eq. (1), we have that Eq. (2) admits a symmetry algebra that is a product of a Heisenberg-Weyl algebra, w_1^c, with a $su(1,1)$ algebra. The subalgebras w_1^c and $su(1,1)$ are the ones associated with coherent states and squeezing.

The time-dependent lowering and raising (ladder) operators in w_1^c are given by

$$A(t) = i\left[\xi(t)\ p - \dot{\xi}(t)\ q\right], \qquad A^\dagger(t) = -i\left[\xi^*(t)\ p - \dot{\xi}^*(t)\ q\right]. \tag{3}$$

Here q and $p = -i\partial_q$ are the ordinary position and momentum operators

$$q = \frac{a + a^\dagger}{\sqrt{2}}, \qquad p = \frac{a - a^\dagger}{i\sqrt{2}}. \tag{4}$$

It is straight forward to demonstrate [1] that the time-dependent functions, ξ and ξ^* in Eq. (3) are two linearly independent, complex solutions of the second-order differential equation[3]

$$\ddot{\gamma} + 2f(t)\gamma = 0. \tag{5}$$

Therefore, the time-dependent functions ξ and ξ^* are determined by the particular Hamiltonian.

[2]The Hamiltonian is $H = -\frac{1}{2}\partial_{qq} + f(t)q^2$.

[3] The 'dot' over a function of t indicates ordinary differentiation with respect to t.

The Wronskian of these solutions is

$$\xi(t)\dot{\xi}^*(t) - \dot{\xi}(t)\xi^*(t) = -i. \tag{6}$$

On account of Wronskian (6), the ladder operators $A(t)$ and $A^\dagger(t)$ satisfy the commutation relation

$$[A(t), A^\dagger(t)] = I, \tag{7}$$

where $I = 1$ is the identity operator.

3 Coherent/squeezed states and uncertainty relations

3.1 Coherent states

First, consider $|\alpha; t\rangle$, the displacement-operator coherent states (DOCS). They are defined by

$$|\alpha; t\rangle = D_A(\alpha)|0; t\rangle = \exp\left[\alpha A^\dagger(t) - \alpha^* A(t)\right]|0; t\rangle. \tag{8}$$

In the above, $D_A(\alpha)$ is the displacement operator and α is a complex constant. Similarly, the equivalent ladder-operator coherent states (LOCS) are defined as

$$A(t)|\alpha; t\rangle = \alpha|\alpha; t\rangle. \tag{9}$$

In Eq. (8), the extremal state $|0; t\rangle$ is a member of a set of number states, $\{|n; t\rangle,\ n = 0, 1, 2, \cdots\}$, which are eigenfunctions of the number operator $A^\dagger(t)A(t)$ [1, 4]:

$$A^\dagger(t)A(t)|n; t\rangle = n|n; t\rangle. \tag{10}$$

An important conceptual point for these time-dependent systems is that the number states, $|n; t\rangle$, are in general NOT eigenstates of the Hamiltonian. That is,

$$H|n; t\rangle = i\partial_t|n; t\rangle \neq (\text{real constant})|n; t\rangle. \tag{11}$$

In particular, the extremal state $|0; t\rangle$ is technically NOT the ground state.

3.2 Uncertainty relations and squeezed states

The operators, A and $A^\dagger$, are linear functionals of q and p. With them, generalized position and momentum operators can be defined:

$$Q = \frac{A + A^\dagger}{\sqrt{2}}, \qquad P = \frac{A - A^\dagger}{i\sqrt{2}}. \tag{12}$$

Their associated commutation relation is

$$[Q, P] = i\mathcal{O} = i[A, A^\dagger], \tag{13}$$

where $\mathcal{O}$ is an Hermitian operator.[4]

Now consider the uncertainty product

$$U(Q,P) = (\Delta Q)^2(\Delta P)^2 = [\langle Q^2\rangle - \langle Q\rangle^2][\langle P^2\rangle - \langle P\rangle^2] \tag{14}$$
$$= [\langle (Q-\langle Q\rangle)^2\rangle][\langle (P-\langle P\rangle)^2\rangle]. \tag{15}$$

Applying the Schwartz inequality to Eq. (15) we have

$$(\Delta Q)^2(\Delta P)^2 \geq \tfrac{1}{4}|i\langle\mathcal{O}\rangle + \langle\{\hat{Q},\hat{P}\}\rangle|^2 \tag{16}$$

$$\geq \tfrac{1}{4}\langle\mathcal{O}\rangle^2 + \tfrac{1}{4}\langle\{\hat{Q},\hat{P}\}\rangle^2 \tag{17}$$
$$\geq \tfrac{1}{4}\langle\mathcal{O}\rangle^2, \tag{18}$$

$\{,\}$ being the anticommutator, and

$$\hat{Q} \equiv Q - \langle Q\rangle, \qquad \hat{P} \equiv P - \langle P\rangle. \tag{19}$$

Eq. (17) is the *Schrödinger Uncertainty Relation* [5]. Equality is satisfied by states $|B,C\rangle$ that are colinear in $\hat{Q}$ and $\hat{P}$:

$$\hat{Q}|B,C\rangle = -i\ B\ \hat{P}|B,C\rangle, \tag{20}$$

where B is a complex constant. Now going to wave-function notation, for what will be the minimum-uncertainty squeezed states (MUSS), Eq. (20) can be rewritten as [6, 7]

$$[Q + iB\ P]\Psi_{ss}(t) = C\Psi_{ss}(t), \tag{21}$$
$$C \equiv \langle Q\rangle + iB\langle P\rangle. \tag{22}$$

In general, B and C are both complex. The solutions to Eq. (21) are "squeezed states" for the system.

B can be understood to be the complex squeeze factor by i) multiplying Eq. (20) on the left first by $\hat{Q}$ and then by $\hat{P}$ and taking the expectation values, ii) then doing the same for the adjoint equation (multiplying on the right), and iii) finally using Eq. (15). This yields

$$B = i\ \frac{(\Delta Q)^2}{\langle\hat{Q}\hat{P}\rangle} = i\ \frac{\langle\hat{P}\hat{Q}\rangle}{(\Delta P)^2}, \tag{23}$$

$$B^* = -i\ \frac{(\Delta Q)^2}{\langle\hat{P}\hat{Q}\rangle} = -i\ \frac{\langle\hat{Q}\hat{P}\rangle}{(\Delta P)^2}, \tag{24}$$

$$|B|^2 = \frac{(\Delta Q)^2}{(\Delta P)^2}. \tag{25}$$

[4] Because of Eq. (7), for the systems studied in this paper $\mathcal{O} = I$. But for now we continue with the general $\mathcal{O}$.

Thus, $|B|$ yields the relative uncertainties of Q and P.

For the particular case $B = 1$, the MUSS are the minimum-uncertainty coherent states (MUCS). These satisfy equality of Eq. (18), the *Heisenberg Uncertainty Relation.* That these are coherent states is easily seen from Eq. (21), which then reduces to $\sqrt{2}$ times the LOCS Eq. (9).

To intuitively see this, consider Eq. (21) for the simple harmonic oscillator. The solutions are

$$\psi_{ss}(q) \propto \exp\left[-\frac{1}{2}\frac{(q-\langle q\rangle)^2}{B} - i\langle p\rangle\ q\right]. \tag{26}$$

B is an extremely complicated functional [8] of the complex parameter λ of the standard su(1,1) squeeze operator:[5]

$$S_a(\lambda(t)) = \exp\left[\tfrac{1}{2}\lambda(t)\ a^\dagger a^\dagger - \tfrac{1}{2}\lambda^*(t)\ aa\right], \qquad \lambda(t) \equiv r(t)e^{i\theta(t)}. \tag{27}$$

The exact relationship depends on if one defines the displacement-operator squeezed states (DOSS) as

$$|\alpha,\lambda;t\rangle = D_A(\alpha)S_a(\lambda)|0\rangle \tag{28}$$

or as $S_a(\lambda)D_A(\alpha)|0\rangle$.

We will now describe and then apply this formalism to a well-known and important system, the Paul trap. The aim is to obtain a deeper insight into the "coherent" and "squeezed" states of this system.

4 The Paul Trap

The Paul trap is a dynamically stable environment for charged particles [9]-[11]. It has been of great use in areas from quantum optics to particle physics.

It's main structure consists of two parts. The first is an annular ring-hyperboloid of revolution, whose symmetry is about the $x-y$ plane at $z=0$. The distance from the origin to the ring-focus of the hyperboloid is r_0. The inner surface of this ring electrode is a time-dependent electrical equipotential surface. The second part of the structure consists of two end-caps. These are hyperboloids of revolution about the z axis. The distance from the origin to the two foci is usually $d_0 = r_0/\sqrt{2}$. The two end-cap surfaces are time-dependent equipotential surfaces with sign opposite to that of the ring. The electric field within this trap is a quadrupole field. With oscillatory potentials are applied, a charged particle *can be* dynamically stable.

Paul gives a delightful mechanical analogy [11]. Think of a mechanical ball put at the center of a saddle surface. With no motion of the surface, it will fall off of the saddle. However, if the saddle surface is rotated *with an appropriate frequency* about the axis

[5] Here, we can allow $\lambda = \lambda(t)$. But note that if, in contrast to the present case, the oprators defining the squeeze operator have time derivatives (e.g., the su(1,1) operators M_- and M_+ of Ref. [2]), then this possibility leads to time-derivative complications and an inequivalent result.

normal to the surface at the inflection point, the particle will be stably confined. The particle is oscillatory about the origin in both the x and y directions. But it's oscillation in the z direction is restricted to be bounded from below by some $z_0 > 0$.

The potential energy can be parametrized as [9]

$$V(x, y, z, t) = V_x(x, t) + V_y(y, t) + V_z(z, t), \tag{29}$$

where

$$V_x(x, t) = +\frac{e}{2r_0^2}\,\mathcal{V}(t)\ x^2 \equiv \frac{1}{2}\Omega_x(t)x^2, \tag{30}$$

$$V_y(y, t) = +\frac{e}{2r_0^2}\,\mathcal{V}(t)\ y^2 \equiv \frac{1}{2}\Omega_y(t)y^2, \tag{31}$$

$$V_z(z, t) = -\frac{e}{r_0^2}\,\mathcal{V}(t)\ z^2 \equiv \frac{1}{2}\Omega_z(t)z^2. \tag{32}$$

In the above,

$$\mathcal{V}(t) = \mathcal{V}_{dc} - \mathcal{V}_{ac}\cos\omega(t - t_0) \tag{33}$$

is the "dc" plus "ac time-dependent" electric potential that is applied between the ring and the end caps. These potentials can be used to solve the classical motion problem. The result is oscillatory Mathieu functions for the bound case [9]. The oscillatory motion goes both positive and negative in the $x - y$ plane, but is constrained to be positive in the z direction.

Exact solutions for the quantum case were first investigated in detail by Combescure [12]. In general, work has concentrated on the z coordinate, but not entirely [13]. Elsewhere [14] we will look at the symmetries, separations of variables, and the number and coherent state solutions of the three-dimensional Paul trap, in both Cartesian and cylindrical coordinates.

5 Coherent physics of the Paul trap

5.1 Coherent states and the classical motion

With the background established in the previous two sections, we want to discuss the coherent/squeezed states of the Paul trap. We focus on the interesting z coordinate and, in particular, use as reference the lovely discussion in Schrade et al. (SMSG) [15]. Using Eq. (32), the Hamiltonian is

$$H = i\partial_t = -\frac{1}{2}\frac{\partial^2}{\partial z^2} + \frac{1}{2}\Omega_z(t)z^2. \tag{34}$$

The connection between the notation of Section 2 and that of SMSG [15] is

$$q = z, \qquad 2f = \Omega_z, \qquad \xi(t) = \tfrac{1}{\sqrt{2}}\epsilon(t). \tag{35}$$

The ladder operators, $A(t)$ and $A^\dagger(t)$, are

$$A(t) = \frac{i}{\sqrt{2}}\left[\epsilon(t)\ p - \dot{\epsilon}(t)\ z\right] = \frac{1}{2}\left\{a[\epsilon(t) - i\dot{\epsilon}(t)] + a^\dagger[-\epsilon(t) - i\dot{\epsilon}(t)]\right\}, \tag{36}$$

$$A^\dagger(t) = -\frac{i}{\sqrt{2}}\left[\epsilon^*(t)\ p - \dot{\epsilon}^*(t)\ z\right] = \frac{1}{2}\left\{a^\dagger[\epsilon^*(t) + i\dot{\epsilon}^*(t)] + a[-\epsilon^*(t) + i\dot{\epsilon}^*(t)]\right\}, \tag{37}$$

Recall that because of Eq. (35), $\epsilon(t)$ is a complex solution to the differential equation (5) and its complex conjugate, $\epsilon^*(t)$, is the other linearly independent solution. The Wronskian of $\epsilon(t)$ and $\epsilon^*(t)$ is a constant:

$$W(\epsilon, \epsilon^*) = \epsilon(t)\dot{\epsilon}^*(t) - \dot{\epsilon}(t)\epsilon^*(t) = -2i. \tag{38}$$

Note from Eq. (32), that Newton's classical equation of motion is

$$F = \ddot{z}_{cl}(t) = -\frac{dV(z,t)}{dz} = \frac{2e}{r_0^2}\ \mathcal{V}(t)\ z \equiv -\Omega_z(t) z_{cl}(t), \tag{39}$$

the solutions being Mathieu functions. But Eq. (35) shows that Eq. (39) is exactly the form of Eq. (5) for $\epsilon(t)$ and $\epsilon^*(t)$. This means that a combination of $\epsilon(t)$ and $\epsilon^*(t)$, up to normalization, follows the classical-motion Mathieu function solutions for $z_{cl}(t)$ and $p_{cl}(t) = \dot{z}_{cl}(t)$. Therefore, we can conveniently and with foresight write these classical solutions in the forms most convenient for the quantum study:

$$z_{cl}(t) = \frac{i}{2}\left\{[\epsilon^*(t)\epsilon_o - \epsilon(t)\epsilon_o^*]\, p_o + [\epsilon(t)\dot{\epsilon}_o^* - \epsilon^*(t)\dot{\epsilon}_o]\, z_o\right\}, \tag{40}$$

$$p_{cl}(t) = \frac{i}{2}\left\{[\dot{\epsilon}^*(t)\epsilon_o - \dot{\epsilon}(t)\epsilon_o^*]\, p_o + [\dot{\epsilon}(t)\dot{\epsilon}_o^* - \dot{\epsilon}^*(t)\dot{\epsilon}_o]\, x_o\right\} = \dot{z}_{cl}(t), \tag{41}$$

where z_o and p_o are initial position and momentum and $\epsilon_o = \epsilon(t_o)$.

Now using Eqs. (12), (36), and (37), with $Q \to Z$,

$$Z(t) = \frac{i}{2}\left\{[\epsilon(t) - \epsilon^*(t)]p - [\dot{\epsilon}(t) - \dot{\epsilon}^*(t)]z\right\} \tag{42}$$

$$
\begin{aligned}
&= \frac{i}{\sqrt{2}}\left\{a\ [(Im\ \dot{\epsilon}(t)) + i(Im\ \epsilon(t))] - a^{\ \dagger}[(Im\ \dot{\epsilon}(t)) - i(Im\ \epsilon(t))]\right\}, \qquad (43)\\
P(t) &= \frac{1}{2}\left\{[\epsilon(t)+\epsilon^*(t)]p - [\dot{\epsilon}(t)+\dot{\epsilon}^*(t)]z\right\} \qquad (44)\\
&= \frac{1}{i\sqrt{2}}\left\{a\ [(Re\ \epsilon(t)) - i(Re\ \dot{\epsilon}(t))] - a^{\ \dagger}[(Re\ \epsilon(t)) + i(Re\ \dot{\epsilon}(t))]\right\}. \qquad (45)
\end{aligned}
$$

It follows that

$$
[Z(t), P(t)] = -\tfrac{1}{2}\left[\epsilon^*(t)\dot{\epsilon}(t) - \epsilon(t)\dot{\epsilon}^*(t)\right] = i, \qquad (46)
$$

where we have used the Wronskian (38).

Using Eqs. (8), (36), and (37), one can calculate that the coherent-state wave functions for the Paul Trap go as [15]

$$
\Psi_{cs}(z,t) \sim \exp\left[-\frac{1}{2}\left(\frac{-i\dot{\epsilon}(t)}{\epsilon(t)}\right)(z - z_{cl}(t))^2 + izp_{cl}(t)\right]. \qquad (47)
$$

The Gaussian form of Ψ can be readily verified [2, 3] by noting that i)

$$
\begin{aligned}
\left(-i\frac{\dot{\epsilon}(t)}{\epsilon(t)}\right) &= \frac{1}{\phi(t)}\left[1 - \tfrac{i}{2}\dot{\phi}(t)\right], \qquad (48)\\
\phi(t) &= \epsilon(t)\epsilon^*(t) \qquad (49)
\end{aligned}
$$

and ii) ϕ is a positive real function of t.[6]

5.2 Uncertainty relations

However, with these wave functions, SMSG [15] found that the $z-p$ Heisenberg Uncertainty Relation is not satisfied. Rather, the $z-p$ Schrödinger Uncertainty Relation is satisfied [1, 15, 16]:

$$
(\Delta z)^2(\Delta p)^2 = \tfrac{1}{4}\left[1 + \tfrac{1}{4}\dot{\phi}^2(t)\right], \qquad (50)
$$

where ϕ is given in Eq. (49).[7]

[6] The function (48) also arises in number-operator states. the relevant equation is $A(t)|0;t\rangle = 0$, when $A(t)$ of Eq. (36) is used. This is the 0-eigenvalue case of Eq. (9).

[7] The right hand side of Eq. (50) is equal to 1/4 for all t only if $\dot{\phi} = 0$; that is, for an harmonic oscillator or a driven oscillator [16].

We observe that not only is this correct but is to be expected. These wave functions were generated by the $Z-P$ variables, and hence should satisfy the Heisenberg uncertainty relation for Z and P. Contrariwise, note that Eq. (36) can be written as

$$A(t)=\frac{1}{2}\left\{a[\epsilon(t)-i\dot{\epsilon}(t)]+a^{\dagger}[-\epsilon(t)-i\dot{\epsilon}(t)]\right\}\equiv \mu a+\nu a^{\dagger}. \tag{51}$$

But with Eq. (38) one has that

$$1=|\mu|^2-|\nu|^2. \tag{52}$$

That is, A and $A^{\dagger}$ must be related to a and $a^{\dagger}$ by a Holstein-Primakoff/Bogoliubov transformation [17] of the form

$$S_a^{-1}(\lambda)\ a\ S_a(\lambda)\ =\ [\cosh r]a+[e^{i\theta}\sinh r]a^{\dagger}, \tag{53}$$

$$S_a^{-1}(\lambda)\ z\ S_a(\lambda)\ =\ \frac{1}{\sqrt{2}}\left\{\left[\cosh r+e^{-i\theta}\sinh r\right]a+\left[\cosh r+e^{i\theta}\sinh r\right]a^{\dagger}\right\}. \tag{54}$$

However, there remains one further complication. The coefficient of a on the right of Eq. (53) is real while that of Eq. (51) is complex. There is a phase offset. This is related to the fact that there are four parameters in Eq. (51) and only two in Eq. (53). That there are only two parameters in Eq (53) follows from the fact that the entire squeezed-state transformation from (z,p) to (Z,P) also involves a displacement. (See Eq. (28).) The displacement supplies the other two parameters, as we now demonstrate.

5.3 Squeezed states

Combine, in the A representation, the DOCS definition of Eq. (8) with the LOCS definition of Eq. (9):

$$A(t)D_A(\alpha)|0;t\rangle=\alpha D_A(\alpha)|0;t\rangle. \tag{55}$$

Next, insert $I=S_a^{-1}(\lambda)\ S_a(\lambda)$ in front of all the operators and multiply on the left by $S_a(\lambda)$ of Eq. (27). Regrouping, we obtain the equation

$$\left(S_aAS_a^{-1}\right)\left(S_aD_A(\alpha)S_a^{-1}\right)S_a(\lambda)\ |0;t\rangle=\alpha\left(S_aD_A(\alpha)S_a^{-1}\right)S_a(\lambda)\ |0;t\rangle. \tag{56}$$

Given Eqs. (51) and (52) and using both the BCH relation [1]

$$\exp\left[\lambda\frac{a^{\dagger}a^{\dagger}}{2}-\lambda^*\frac{aa}{2}\right]=\exp\left[\gamma_+\frac{a^{\dagger}a^{\dagger}}{2}\right]\exp\left[\gamma_3(a^{\dagger}a+\tfrac{1}{2})\right]\exp\left[\gamma_-\frac{aa}{2},\right], \tag{57}$$

$$\gamma_-=-e^{-i\theta}\tanh r,\qquad \gamma_+=e^{i\theta}\tanh r,\qquad \gamma_3=\ln\cosh r, \tag{58}$$

and also the theorem [18]

$$\exp[X]\exp[Y]\exp[-X] = \exp[e^X Y e^{-X}], \tag{59}$$

it follows that

$$S_a(\lambda)D_A(\alpha)S_a^{-1}(\lambda) = \exp\left[\beta a^\dagger - \beta^* a\right] \equiv D_a(\beta), \tag{60}$$

$$\beta = \alpha v^* - \alpha^* u, \qquad \beta^* = \alpha^* v - \alpha u^*. \tag{61}$$

In addition, we see that

$$S_a(\lambda)AS_a^{-1}(\lambda) = va + ua^\dagger, \tag{62}$$

where the coefficients u and v are

$$u = \nu\cosh r - \mu e^{i\theta}\sinh r, \qquad v = \mu\cosh r - \nu e^{-i\theta}\sinh r. \tag{63}$$

Note that u and v are functions of t because μ and ν are. Furthermore, we have

$$|v|^2 - |u|^2 = |\mu|^2 - |\nu|^2 = 1. \tag{64}$$

Therefore, Eq. (56) becomes [8]

$$\left\{va + ua^\dagger\right\}\; D_a(\beta)\; S_a(\lambda)\; |0;t\rangle = \{\alpha(\lambda,\beta)\}\; D_a(\beta)\; S_a(\lambda)\; |0;t\rangle = \{\alpha(\lambda,\beta)\}\; |\alpha(\lambda,\beta),t\rangle. \tag{65}$$

This satisfies both the displacement-operator definition of squeezed states (to the right of the brackets) and the ladder operator definition of squeezed states (the curly brackets) [2]. This completes the transformation of a coherent state in the $(A, A^\dagger)$ representation into a squeezed state in the $(a, a^\dagger)$ representation. Since the squeeze operator is invertible, we can obviously reverse this process. [9]

6 Conclusion

The coherent states of (Z, P) or $(A, A^\dagger)$ are squeezed states of (z, p) or $(a, a^\dagger)$, and *vice versa*. This makes sense. The fundamental potentials are of different widths, so their coherent-state Gaussians are also of different widths. Indeed, Eq. (47) shows this. For the (z, p) uncertainty relation, $[-i\dot{\epsilon}(t)/\epsilon(t)]$ is a squeeze factor.

[8] In Eq. (65), α is indeed the quantity of Eq. (55). However, because of our construction, it can be obtained from Eqs. (61) and (63) as a functional of λ and β.

[9] Because of the isomorphism of the two Heisenberg-Weyl algebras, $\{a, a^\dagger, I\}$ and $\{A(t), a^\dagger(t), I\}$, the converse of this result follows. Starting with a coherent-state equation in the $(a, a^\dagger)$ representation: $aD_a(\eta)|0\rangle = \eta D_a(\eta)|0\rangle$, where η is complex, we can map this equation into an analogous squeezed-state equation in the $(A, A^\dagger)$ representation with an appropriate squeezing operator in the $(A, A^\dagger)$ representation. This is done going through an analogous procedure as above with the aid of the equations $a = \rho A + \sigma A^\dagger$, $\rho = \frac{1}{2}\left(\epsilon^*(t) - i\dot{\epsilon}^*(t)\right)$, and $\sigma = \frac{1}{2}\left(\epsilon(t) - i\dot{\epsilon}(t)\right)$ and the condition $|\sigma|^2 - |\rho|^2 = 1$.

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